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$\mathfrak{Algebraic\ Number\ Theory}$

Instructor: Ramdin Mawia **Time:** December 30, 2022; 10:00–13:00.

INSTRUCTIONS

i. Attempt THREE problems. Each question carries 20 marks. The maximum you can score is **45**.

ii. MMath students should attempt at least one problem from each of Group I and Group II.

iii. BMath students can choose any of the problems.

iv. You may use any of the results proved in class, unless you are asked to prove or justify the result itself. You may also use results from other problems in this question paper, provided you attempt and correctly solve the problem.

Group I

1. Let $E/K, L/K$ be extensions of number fields. Prove that a prime $\mathfrak{p}$ of K splits completely in the composite EL if and only if it splits completely in E and L.

2. Let L/K be a Galois extension of number fields, and let $\mathfrak{P}$ be a prime of L lying above a prime $\mathfrak{p}$ of K. Prove that there is a surjective group homomorphism $G_{\mathfrak{P}} \to \mathrm{Gal}(k(\mathfrak{P})/k(\mathfrak{p}))$. Here $k(\mathfrak{P}) = \mathcal{O}_L/\mathfrak{P}$ and $k(\mathfrak{p}) = \mathcal{O}_K/\mathfrak{p}$. Deduce that if $\mathrm{Gal}(L/K)$ is not cyclic, then no prime of K remains inert in L.

3. Define the *absolute discriminant* Δ_K of a number field K and prove that it is congruent to either 0 or 1 mod 4. Determine the ring of integers in $\mathbb{Q}[\alpha]$ where $\alpha^3 + 4\alpha + 3 = 0$.

4. State true or false, **without** justifications:

   (a) The splitting field of the polynomial $X^3 + 4X + 3$ over $\mathbb{Q}$ is a cyclotomic extension of $\mathbb{Q}$.

   (b) If $\mathfrak{p}$ is a maximal ideal of a the ring of integers $\mathcal{O}_K$ of a number field K, then $\mathcal{O}_K/\mathfrak{p}$ is a finite field.

   (c) 2021 is a square modulo 2023.

   (d) The ring of integers in $\mathbb{Q}[\sqrt{5}]$ is $\mathbb{Z}[\sqrt{5}]$.

   (e) The prime 29 is inert in the field $\mathbb{Q}[\alpha]$ where $\alpha^3 + \alpha + 3 = 0$.

   (f) Let A be a Dedekind domain with field of fractions K, and let B be the integral closure of A in a finite separable extension L of K. Then a maximal ideal $\mathfrak{P}$ of B is ramified over K if and only if it is contained in the different $\mathfrak{D}_{B/A}$.

   (g) Let A be a Dedekind domain with field of fractions K, and let B be the integral closure of A in a finite separable extension L of K. Then $\mathrm{N}^L_K \Delta_{B/A} = \mathfrak{D}_{B/A}$, i.e., the different is the norm of the discriminant.

   (h) The integral closure of a DVR A in a finite separable extension of its field of fractions is free as an A-module.

   (i) There is a Dedekind domain one of whose ideals cannot be generated by fewer than 3 elements.

   (j) A Dedekind domain which has only finitely many maximal ideals is a PID.

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5. Let $(K, |\cdot|)$ be a local field, and let V be a finite-dimensional K-vector space. Prove that any two norms on V are equivalent.

6. Let $(K, |\cdot|)$ be a valued field. Prove that it is a local field if and only if (i) $(K, |\cdot|)$ is complete, (ii) $|\cdot|$ is discrete and (iii) the residue field $\overline{K}$ is finite.

7. Prove that the field of rational numbers $\mathbb{Q}$ is not complete with respect to any nontrivial absolute value.

8. State true or false, **without** justifications:

   (a) The field of Laurent series $\mathbb{F}_p((T))$ in one variable T with the T-adic valuation $v_T(T) = 1$ is a local field.

   (b) Every local field is complete.

   (c) Every complete field is local.

   (d) The ring of p-adic integers $\mathbb{Z}_p$ is a connected topological space.

   (e) If $(K, |\cdot|)$ is a local field, then $|\cdot|$ has at least $|\overline{K}|$ inequivalent extensions to any finite extension L of K.